STABILITY OF COMBUSTION OF POWDER IN A PHENOMENOLOGICAL MODEL WITH NONADIABATIC FLAME

Yu. A. Gostintsev, P. F. Pokhil,* and L. A. Sukhanov

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A stability criterion for combustion of powder is obtained, taking into account the effect of the processes in the gas phase. It is shown that consideration of the effects of a nonadiabatic flame leads to the stability reserve of combustion being reduced and the natural frequency of vibrations being lowered. The effects thus found are physically explained by the radiation of a part of energy from the combustion zone with thermal and acoustic waves.

The internal instability of stationary combustion of powder was considered in [1] neglecting effects connected with the existence of the flame and the presence of a feedback between the perturbations of pressure p in the gas and the velocity of combustion u_s.

A stability criterion for combustion was found, and the value of the natural frequency of the heated layer of the condensed phase (k-phase) was established.

Here, within the limits of the phenomenological theory of nonstationary combustion of powder with nonadiabatic flame (nonadiabatic is understood in the sense that a flow of heat from the flame in the k-phase takes place and that the heat output in the flame depends on the pressure [2]), we analyze the stability of combustion with the influence of the gas phase taken into account. The general system of equations describing the process, in the moving coordinate system connected with the combustion surface, has the form [3]

$$\frac{\partial T}{\partial t} = \varkappa_s \frac{\partial^2 T}{\partial x^2} - u_s \frac{\partial T}{\partial x} \quad (-\infty < x \leqslant 0)$$
$$T = T_s \quad \text{for} \quad x = 0, \quad T \to T_0 \quad \text{for} \quad x \to -\infty$$
$$T = T_0 + (T_s^\circ - T_0)\exp(u_s^\circ x/\varkappa_s), \quad T_s = T_s^\circ, \quad u_s = u_s^\circ, \tag{1}$$
$$T_F = T_F^\circ \quad \text{for} \quad t = 0$$
$$u_s = u_s(p, \varphi_s), \quad T_s = T_s(p, \varphi_s) \quad (\varphi_s = (\partial T/\partial x)_0)$$

for the k-phase; for the gas above the flame ($\infty > x \geq 0$) it has the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho u = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad p = \rho R T_g$$
$$c_p \rho\left(\frac{\partial T_g}{\partial t} + u\frac{\partial T_g}{\partial x}\right) = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}, \quad \frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x}(\rho h u) = 0$$
$$T_F = T_F(p, \varphi_s), \quad h = h(p, \varphi_s), \tag{2}$$
$$(\rho u)_s = (\rho u)_F = \rho(u_s + u_g) \quad \text{for} \quad x = 0$$
$$p = p^\circ, \quad \rho = \rho^\circ, \quad u^\circ = u_s^\circ + u_g^\circ, \quad T_g = T_F^\circ, \quad h = h_F^\circ \quad \text{for} \quad t = 0$$

*Deceased.

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When writing the equations for the gas phase, we assume that chemical reactions are absent behind the flame and the products are chemically "frozen" (h is the chemical enthalpy of the combustion products).

Let one of the parameters determining the combustion in the gas or in the k-phase receive a small deviation Δ from its stationary value.

We shall investigate the stability of the stationary solution of the system (1), (2). We introduce the dimensionless quantities

$$\pi = p / p^\circ, \quad w = u / u^\circ, \quad \vartheta_g = T_g / T_F^\circ, \quad i_x = h / h^\circ, \quad M = u^\circ / c^\circ \; (c^{\circ 2} = \gamma p^\circ / \rho^\circ)$$

Eliminating density from (2), we obtain the following linearized system of equations, in terms of the combustion products, for the perturbations:

$$\begin{gathered}
\frac{\partial \delta\pi}{\partial t} + u^\circ \frac{\partial \delta\pi}{\partial x} + u^\circ \frac{\partial \delta w}{\partial x} - \frac{\partial \delta\vartheta_g}{\partial t} - u^\circ \frac{\partial \delta\vartheta_g}{\partial x} = 0 \\
\frac{\partial \delta w}{\partial t} + u^\circ \frac{\partial \delta w}{\partial x} + \frac{c^{\circ 2}}{\gamma u^\circ} \frac{\partial \delta\pi}{\partial x} = 0 \\
\frac{\partial}{\partial t} \delta\vartheta_g + u^\circ \frac{\partial}{\partial x} \delta\vartheta_g - \frac{\gamma - 1}{\gamma}\left(\frac{\partial \delta\pi}{\partial t} + u^\circ \frac{\partial \delta\pi}{\partial x}\right) = 0 \\
\frac{\partial}{\partial t} \delta i_x + u^\circ \frac{\partial}{\partial x} \delta i_x = 0
\end{gathered} \tag{3}$$

We seek the solution of (3) in the form $\delta z_j = \Delta Z_j \exp i\omega t$. Then

$$\begin{gathered}
u^\circ \frac{d\Pi}{dx} = \frac{i\omega M^2}{M^2 - 1}(\gamma W - \Pi), \quad u^\circ \frac{dW}{dx} = \frac{i\omega}{M^2 - 1}\left(\frac{\Pi}{\gamma} - M^2 W\right) \\
u^\circ \frac{d\Theta}{dx} = \frac{i\omega(\gamma - 1)}{\gamma(M^2 - 1)}(M^2 \gamma W - \Pi) - i\omega\Theta, \quad u^\circ \frac{dI_x}{dx} = -i\omega I_x
\end{gathered} \tag{4}$$

The particular solutions (4) are represented by waves of the form $Z_j = |Z_j| \exp k_j x$. To determine the amplitude $|Z_j|$ and the wave vectors k_j, we have the system of equations

$$\begin{gathered}
|\Pi|\left(u^\circ k + \frac{M^2\omega}{M^2 - 1}\right) - \frac{M^2\gamma\omega}{M^2 - 1}|W| = 0, \quad |I_x|(ku^\circ + \omega) = 0 \\
-|\Pi|\frac{\omega}{\gamma(M^2 - 1)} + |W|\left(ku^\circ + \frac{M^2\omega}{M^2 - 1}\right) = 0 \\
|W|\frac{(\gamma - 1)M^2\omega}{M^2 - 1} - |\Theta|(ku^\circ + \omega) - |\Pi|\frac{(\gamma - 1)\omega}{\gamma(M^2 - 1)} = 0
\end{gathered} \tag{5}$$

Solving the characteristic determinant (5) we find

$$k_1 = -\frac{\omega}{c^\circ + u^\circ}, \quad k_2 = \frac{\omega}{c^\circ - u^\circ}, \quad k_3 = k_4 = -\frac{\omega}{u^\circ} \tag{6}$$

Here k_1 is the wave vector of the outgoing wave, k_2 is the wave vector of the incident wave, while k_3 and k_4 are the wave vectors of the induced waves of temperature and chemical enthalpy.

With (6) taken into account, the general solution of (3) is written in the form

$$\begin{gathered}
\delta\pi = \Delta\gamma M e^{i\omega t}(C_1 e^{ik_1 x} - C_2 e^{ik_2 x}) \\
\delta w = \Delta e^{i\omega t}(C_1 e^{ik_1 x} + C_2 e^{ik_2 x}) \\
\delta\vartheta_g = \Delta(\gamma - 1)M e^{i\omega t}(C_1 e^{ik_1 x} - C_2 e^{ik_2 x}) + \Delta C_3 e^{i\omega t + ik_3 x} \\
\delta i_x = \Delta C_4 e^{i\omega t + ik_4 x}
\end{gathered} \tag{7}$$

Since the incident wave is absent (we consider only the internal stability of the process), $C_2 = 0$ and the general solution (7) must contain only waves departing from the combustion surface. It has the form (the wave of chemical enthalpy is expressed independently; it does not influence stability and is not subsequently considered)

$$\begin{gathered}
\delta\pi = \Delta\gamma M C_1 e^{i\omega t + ik_1 x}, \quad \delta w = \Delta C_1 e^{i\omega t + ik_1 x} \\
\delta\vartheta_g = \Delta(\gamma - 1) M C_1 e^{i\omega t + ik_1 x} + \Delta C_3 e^{i\omega t + ik_3 x}
\end{gathered} \tag{8}$$

We introduce new dimensionless variables which are more convenient for the solution of the system (1) which properly describes the combustion of powder:

$$\xi = \frac{u_s^\circ}{\varkappa_s} x, \quad \tau = \frac{u_s^{\circ 2}}{\varkappa_s} t, \quad \vartheta = \frac{T - T_0}{T_s^\circ - T_0}, \quad v = \frac{u_s}{u_s^\circ}, \quad \Omega = \frac{\varkappa_s}{u_s^{\circ 2}} \omega$$

$$\zeta_{1,2,3} = \frac{\varkappa_s}{u_s^\circ} k_{1,2,3}, \quad \varphi = \frac{\varphi_s}{\varphi_s^\circ}$$

Then (8) assumes the form

$$\begin{aligned} \delta\pi &= \Delta |\Pi| e^{i\Omega\tau + i\zeta_1\xi} \quad (|\Pi| = \gamma M C_1) \\ \delta w &= \frac{\Delta |\Pi|}{\gamma M} e^{i\Omega\tau + i\zeta_1\xi} \\ \delta\vartheta_g &= \frac{\gamma - 1}{\gamma} \delta\pi + \Delta C_3 e^{i\Omega\tau + i\zeta_3\xi} \end{aligned} \tag{9}$$

Linearizing (1), we obtain

$$\begin{aligned} &\frac{\partial^2}{\partial\xi^2} \delta\vartheta - \frac{\partial}{\partial\xi} \delta\vartheta - \frac{\partial}{\partial\tau} \delta\vartheta = e^{\xi} \delta v \\ &\delta\vartheta(\xi \to \infty) = 0, \quad \delta\vartheta(\xi = 0, \tau) = \delta\vartheta_s, \quad \delta\vartheta(\xi, \tau = 0) = 0 \\ &\delta\vartheta_s = \left(\frac{\partial\vartheta_s}{\partial\pi}\right)_\varphi \delta\pi + \left(\frac{\partial\vartheta_s}{\partial\varphi}\right)_\pi \delta\varphi \\ &\delta\vartheta_F = \left(\frac{\partial\vartheta_F}{\partial\pi}\right)_\varphi \delta\pi + \left(\frac{\partial\vartheta_F}{\partial\varphi}\right)_\pi \delta\varphi \\ &\delta v = \left(\frac{\partial v}{\partial\pi}\right)_\varphi \delta\pi + \left(\frac{\partial v}{\partial\varphi}\right)_\pi \delta\varphi \end{aligned} \tag{10}$$

Since the pressure on the combustion surface varies according to the law $\delta\pi \sim \exp i\Omega\tau$, the solutions (10) must be sought amongst functions of the same form. According to [2], for the complex amplitudes we can find

$$\begin{aligned} &V = V_1 |\Pi|, \quad V_1 = \frac{\nu + \delta(\alpha - 1)}{1 - k + (\alpha - 1)(r - ik/\Omega)} \\ &\varepsilon\Theta_F = |\Pi| \left(s - \nu\frac{q}{k} + \frac{q}{k} V_1\right), \quad \alpha = 1/2\left(1 + \sqrt{1 + 4i\Omega}\right) \end{aligned} \tag{11}$$

Here

$$\nu = \left(\frac{\partial \ln u_s^\circ}{\partial \ln p}\right)_{T_0}, \quad k = (T_s^\circ - T_0)\left(\frac{\partial \ln u_s^\circ}{\partial T_0}\right)_p, \quad r = \left(\frac{\partial T_s^\circ}{\partial T_0}\right)_p$$

$$\mu = \left(\frac{\partial T_s^\circ}{\partial \ln p}\right)_{T_0} (T_s^\circ - T_0)^{-1}, \quad s = \left(\frac{\partial \ln T_F^\circ}{\partial \ln p}\right)_{T_0},$$

$$q = (T_s^\circ - T_0)\left(\frac{\partial \ln T_F^\circ}{\partial T_0}\right)_p$$

$$\delta = \nu r - \mu k, \quad \varepsilon = (T_s^\circ - T_0) / T_F^\circ$$

The parameters ν, k, r, μ, s, q, and δ define the properties of the reaction zones in the flame in the k-phase of the powder.

The constants $|\Pi|$ and C_3 in (9) and (11) are found from the condition that the solutions are "sewn together" on the flame:

$$\delta\vartheta_g = \delta\vartheta_F, \quad \delta w_F = \delta w_g$$

With the equations of state and conservation of mass taken into account, we find

$$\begin{aligned} &|\Pi| \left(1 - \nu\frac{q}{k} + \frac{q}{k} V_1 - \frac{\gamma - 1}{\gamma}\right) - C_3 = 0 \\ &|\Pi| \left[V_1\left(1 + \frac{q}{k}\right) + s - \nu\frac{q}{k} - 1 - \frac{1}{\gamma M}\right] = 0 \end{aligned} \tag{12}$$

The characteristic equation

$$\begin{bmatrix} s - \nu\frac{q}{k} + \frac{q}{k} V_1 - \frac{\gamma - 1}{\gamma}, & -1 \\ V_1\left(1 + \frac{q}{k}\right) + s - \nu\frac{q}{k} - 1 - \frac{1}{\gamma M}, & 0 \end{bmatrix} = 0$$

follows from (12).

Using the expression V_1 from (11), we find from this equation that

$$\left(1 - s + \nu \frac{q}{k} + \frac{1}{\gamma M}\right) \frac{1}{1 + q/k} = \frac{\nu + \delta(\alpha - 1)}{1 - k + (\alpha - 1)(r - ik/\Omega)} \tag{13}$$

Since $\Omega = \mathrm{Re}\,\Omega + i\mathrm{Im}\,\Omega$ and the solution of the problem was sought in the form $\sim \exp i\Omega\tau = \exp[i(\mathrm{Re}\Omega)\tau - (\mathrm{Im}\,\Omega)\tau]$, the stability condition of combustion is equivalent to the inequality $\mathrm{Im}\,\Omega \geq 0$ being satisfied.

We shall investigate the characteristic equation (13).

Denoting

$$\sigma = i\Omega \quad (\alpha = {}^1/_2(1 + \sqrt{1 + 4\sigma})), \quad a_1 = (1 - k)\,a_0 - \nu\gamma M$$
$$a_2 = ra_0 - \delta\gamma M, \; a_3 = a_0 k, \; a_0 = [1 + \gamma M (1 - s + \nu q / k)]\, k / (k + q)$$

we rewrite (13) in the form

$$(\sqrt{1 + 4\sigma} - 1)(a_2 / a_3 + 1 / \sigma) = -2a_1 / a_3 \tag{14}$$

The quantity σ in Eq. (14) in the general case is complex ($\sigma = x + iy$) (x and y are real numbers); therefore (14) can be brought into the form

$$a_2^2\sigma^2 + \sigma\,(2a_2a_3 + a_1a_3 - a_1^2) + a_3^2 + a_1a_3 = 0 \tag{15}$$

After separating the real and imaginary parts, we find

$$x = a_2^2 / 2\,(2a_2a_5 + a_1a_2 - a_1^2) \tag{16}$$

$$a_2^2y^2 = a_3^2 + a_1a_3 - a_2^2\,(2a_2a_3 + a_1a_2 - a_1^2)^2 / 4 \tag{17}$$

from (15).

By the definition the value of y is real. We find a region of parameters where this requirement is fulfilled. For this we consider the sign of the inequality $(a_3^2 + a_1a_3) - a_2^2(2a_2a_3 + a_1a_2 - a_1^2)^2/4$. If it is greater than zero, then y is a real number, while in the contrary case it is purely imaginary.

An analysis shows that the left side of the inequality is negative if

$$(1 - k)\frac{k}{k + q}\left[\left(1 - s + \nu\frac{q}{k}\right)\gamma M + 1\right] - \nu\gamma M \geqslant 0 \tag{18}$$

or

$$k \leqslant 1 - \frac{\nu(1 + k)\gamma M}{(1 - s)\gamma M + 1} = k_0$$

Thus in the region $k < k_0$ the characteristic equation (14) has no complex roots, and y is a purely imaginary number. This contradicts its definition. Therefore in the region $k < k_0$ the roots of the characteristic equation are real, or they do not exist at all. Assuming σ to be real and positive (physically this corresponds to σ being defined as the perturbation frequency), we shall consider the solution (14) in the region $k < k_0$.

We write

$$\frac{a_1}{a_3} = \frac{k + q}{k^2}\left\{\frac{k(1 - k)}{k + q}\left[\left(1 - s + \nu\frac{q}{k}\right)\gamma M + 1\right] - \nu\gamma M\right\}\left[\left(1 - s + \frac{\nu q}{k}\right)\gamma M + 1\right]^{-1}$$

$$\frac{a_2}{a_3} = \left\{\frac{rk}{k + q}\left[\left(1 - s + \frac{\nu q}{k}\right)\gamma M + 1\right] - \delta\gamma M\right\}\frac{(k + q)}{k^2}\left[\left(1 + s + \frac{\nu q}{k}\right)\gamma M + 1\right]^{-1}$$

We see that the left side of (14) for $k < k_0$ is negative ($a_1/a_3 > 0$ in view of (18) and the denominator being positive). The right side of (14) is always positive if the inequality $k > \nu r - r/\gamma M - (1 - s)r - \mu k = k_0'$ is satisfied.

In the case of admissible values of the parameters the quantity k_0' is always negative and, consequently, in the region of values $k_0' < k < k_0$ the right and left sides of the characteristic equation have opposite signs. This signifies the absence of real roots.

Thus, in the case $k < k_0$ Eq. (14) has no roots and combustion is stable. Here the stability criterion assumes the form

$$k < k_0 = 1 - \nu (1 + q)\gamma M / [1 + (1 - s) \gamma M] \tag{19}$$

We now investigate the solution of the characteristic equation (14) in the complex region ($k > k_0$):

$$\sigma = (i\Omega)_{1,2} = \frac{1}{2}\left\{\left(\frac{a_1}{a_2}\right)^2 - \frac{a_1}{a_2} - 2\frac{a_3}{a_2} \pm \sqrt{\left[\frac{a_1}{a_2} + 2\frac{a_3}{a_2} - \left(\frac{a_1}{a_2}\right)^2\right]^2 - \frac{4(a_3^2 + a_1 a_3)}{a_2^2}}\right\} \tag{20}$$

According to the definition of stability (Im $\Omega \geq 0$) we obtain the criterion

$$(a_1/a_2)^2 - a_1/a_2 - 2a_3/a_2 \leqslant 0$$

Since under the usual conditions $a_2 > 0$, we have $a_1^2/a_2 - a_1 - 2a_3 \leq 0$.

Substituting here the values a_0, a_1, a_2 and a_3, we can find the stability condition for $k > k_0$. Finally the stability condition of combustion of powder, with the effect of the gas phase taken into account, is written in the form for

$$k < 1 - \nu (1 + q)\gamma M \; [(1 - s)\gamma M + 1]^{-1} = k_0 \; :$$

combustion is always stable; for $k > k_0$ combustion is stable only if

$$r\frac{k}{k+q}\left[\left(1 - s + \frac{\nu q}{k}\right)\gamma M + 1\right] - \delta\gamma M \geqslant \left\{(1-k)\frac{k}{k+q}\left[\left(1 - s + \frac{\nu q}{k}\right)\gamma M + 1\right] - \nu\gamma M\right\}^2 \times$$
$$\times \left[\frac{(1+k)k}{k+q}\left[\left(1 - s + \frac{\nu q}{k}\right)\gamma M + 1\right] - \nu\gamma M\right]^{-1} \tag{21}$$

The natural frequency of thermal oscillations of the combustion zone of powder on the boundary of stability can be determined from (20):

$$\Omega^* = \left\{\frac{r}{k+q}\left[\left(1 - s + \frac{\nu q}{k}\right)\gamma M + 1\right] - \delta\gamma M\right\}^{-1} \times$$
$$\times \sqrt{(k+q)^{-1}\left[1 + \left(1 - s + \frac{\nu q}{k}\right)\gamma M\right]\left\{\frac{k}{k+q}\left[1 + \left(1 - s + \frac{\nu q}{k}\right)\gamma M\right] - \nu\gamma M\right\}} \tag{22}$$

In the case of combustion of powders under the usual conditions the Mach number is small. Therefore, expanding (19), (21) and (22) in a series of γM, we find that combustion is stable always if $k < 1 - \nu\gamma M(1 + q) = k_0$; for $k > k_0$ it is stable only if

$$r \geqslant \frac{(k-1)^2}{k+1} - (k+q)\,\gamma M\left(\mu + \frac{\nu(1-k)(3+k)}{(1+k)^2}\right)$$

Here

$$\Omega^* = \frac{\sqrt{k}}{r} + \frac{k+q}{2r\sqrt{k}}\left(-\nu + \frac{2\delta\sqrt{k}}{r}\right)\gamma M$$

The results of an investigation of combustion stability obtained in [1, 4] follow from the expressions presented above, if we neglect the effect of the gas phase (γM = 0).

We see that the effect of the gas phase leads to a decrease both in the stability reserve of combustion and in the natural frequency of the powder. Physically this is connected with the radiation of a part of energy from the combustion zone with acoustic and thermal waves [2].

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